Talent Supply, the Contest Success Function, and the Invariance Proposition

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Abstract: Rottenberg's invariance proposition has been shown to depend on talent supply elasticity ("closed" versus "open" leagues). But this finding relies on (perhaps) needlessly restrictive assumptions about the marginal product of talent. We extend these results to the general case of the two-team league of profit maximizing owners for both equal proportion gate and modern pooled revenue sharing. We also show that, in addition to talent supply elasticity, the invariance proposition also depends on the form of the underlying contest success function. In addition, a commonly-used talent measurement assumption is a much stronger assumption than previously recognized.

Keywords: talent supply, contest success, sports, invariance principle

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I. Introduction

Rottenberg's (1956) original "invariance proposition" (IP) has been used to argue that the distribution of talent in a sports league is invariant with respect to revenue sharing, player free agency, and player drafts. A long line of the formal theoretical literature based on a competitive talent market found support for the IP (El Hodiri and Quirk, 1971; Quirk and El Hodiri, 1974; Heilmann and Wendling, 1976; Quirk and Fort, 1992; Fort and Quirk, 1995; Vrooman, 1995). Others have rigorously shown that support for the IP varies in this model by owner objective function (Kesenne, 1996, 1999, 2006; Rasher, 1997), whether fans care about relative or absolute talent (Marburger, 1997), and the relative size of revenue function shifts with respect to team quality in smaller- and larger-revenue markets (Kesenne, 2005).

More recently, theoretical work on the IP explores the importance of talent supply elasticity (TSE) assumptions in this past work. The perfectly inelastic extreme (TSE = 0) is typically referred to as the "closed" league. The stock of talent is fixed and reallocation between teams is zero-sum. Should one team owner wish to alter the quality of their team by changing their player roster the only source of talent is the rosters of other teams in the league. At the other perfectly elastic extreme ($\text{TSE} = \infty$) lies the "open" league; total talent is easily augmented by similar quality players from other leagues. Each team in an open league can simultaneously increase talent without any effect on talent on other teams.

This distinction is well-known and the stuff of textbook treatments (Dobson and Goddard, 2001, Chapter 3). However, Rascher (1997), Szymanski (2003, 2004), Szymanski and Kesenne (2004, Appendix 2), Easton and Rockerbie (2005), and Kesenne (2006) argue that assumptions about TSE also dictate differences in owner beliefs about the impacts of their talent choices on each other. For example, Szymanski (2004) implies that the closed league cannot have Nash equilibrium because, by definition, Nash conjectures are that one team owner's choice of talent has no impact on the choice of talent by other team owners. To us, this direction in the literature is more about reconciling Nash equilibrium in *both* closed or open markets rather than excluding particular league forms from Nash consideration (Fort and Winfree, 2008a).

In any event, much like Fort and Quirk (2007), we choose to push the traditional competitive talent market model in new directions rather than abandon it outright. We delve into the important underpinnings of the model, namely, the interaction between TSE and the contest success function (CSF). Fort and Winfree (2008b) show that assumptions about TSE and the CSF actually determine the marginal product of talent (MPT) and that some of these assumptions put equilibrium stability in jeopardy. Here, we show that, at the most general modeling level, in the competitive talent market model, the IP holds for closed leagues but not for open leagues. Further, and also generally, the IP holds for a popular assumption in the competitive talent market literature that sets MPT equal to unity (henceforth, the unit marginal product convention, or UMPC). Since both of these findings hold generally, they are important extensions since the previous literature either ignores the CSF altogether or assumes a restricted version of the logistic CSF. Interestingly, following some encouragements by Skaperdas (1996) to explore various CSFs, we find an additional, and novel, result. The IP does indeed hold for *open* *leagues* under a particular linear CSF. All of our findings extend to modern pooled sharing as well as equal proportion gate sharing.

The paper proceeds as follows. In Section II, we set forth the elements underlying our analysis—the relationship between MPT and TSE; the UMPC; particular CSFs for subsequent analysis; and the profit-maximizing equilibrium. Our general findings on the IP are in Section III while specific findings for the logistic CSF are in Section IV. Our findings for the special linear CSF, including that the IP holds for open leagues, are in Section V. Conclusions round out the paper. Generally, closed leagues exhibit the IP while open leagues do not, the UMPC is a stronger assumption than previously recognized, but whether a given league exhibits the IP depends on more than just TSE.

II. Elements of the Analysis

In this section, we detail the elements of our subsequent analysis. First, we explore the relationship between MPT and TSE (closed and open leagues). Then we specify the UMPC and our chosen CSFs for subsequent analysis. The section concludes with our optimization foundation and a description of the general conditions required in order for the IP to hold. The subsequent sections then assess the presence of this condition generally and for the CSFs specified in this section for both closed and open leagues.

With two teams, the general idea of the CSF for the production of winning percents is just $w_1 = w_1(t_1, t_2)$ and $w_2 = w_2(t_1, t_2)$, where winning percent w_1 for team 1 and w_2 for team 2 depend on the talent choices by both owners, (t_1, t_2). The adding-up constraint for league play requires the sum of winning percents to equal half the number of teams in the league. This is easy to see from the definition of the average for an n-team league, $\frac{1}{n}\sum_{i=1}^{n} w_i = 0.500$ and multiplying by n gives $\sum_{i=1}^{n} w_i = 0.5n$. In the two-team league case, the adding-up constraint is just $w_2 = 1 - w_1$.

The derivative of this general CSF with respect to own talent is MPT for each team:

$$\text{(1)} \quad MPT_1 = \frac{dw_1}{dt_1} = \frac{\partial w_1}{\partial t_1} + \frac{\partial w_1}{\partial t_2}\frac{dt_2}{dt_1}, \; MPT_2 = \frac{dw_2}{dt_2} = \frac{\partial w_2}{\partial t_2} + \frac{\partial w_2}{\partial t_1}\frac{dt_1}{dt_2}.$$

This demonstrates that MPT is quite different from the marginal "physical" product in the production of most goods and services because winning is partly determined by $\frac{dt_j}{dt_i}$, i = 1, 2. It is here that TSE enters into the modeling problem. For a closed league, $TSE = 0$ (perfectly inelastic) and talent can increase for one owner only at the expense of another owner. This dictates that $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2} = -1$. For an open league, $TSE = \infty$ (perfectly elastic) and talent can increase for one owner with no impact at all on another owner. This dictates that $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2} = 0$.

The next element of our analysis is the UMPC. Under the UMPC, talent is measured in units such that a one unit increase in talent increases winning percent by one unit, that is, MPT = 1 so that $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2} = 1$. The UMPC has been commonly used in analyzing sports leagues from Quirk and Fort (1992), to Fort and Quirk (1995), indirectly in Vrooman (1995), and on to Kesenne (2000, 2005). However, its separate implications for equilibrium outcomes like the IP have not been examined in past work and we remedy that in subsequent sections.

In addition to treating the general case of any CSF, we analyze two particular CSFs. The first is the general logistic CSF:

$$(2)\quad w_1 = \frac{t_1^{\gamma}}{t_1^{\gamma} + t_2^{\gamma}},\ w_1 = \frac{t_2^{\gamma}}{t_1^{\gamma} + t_2^{\gamma}},\ w_2 = 1 - w_1.$$

This CSF is well-known in the literature on tournaments, contests, and rent-seeking (see the review in Corchon, 2007). Szymanski (2003) covers its background and history, especially in the sports economics literature. Skaperdas (1996) shows that an attractive set of analytical axioms are uniquely satisfied by the CSF in (2). However, Skaperdas also points out that empirical support for this particular CSF in real-world contests is extremely limited so that adding other CSFs to the mix for the sake of empirical analysis is worthwhile (p. 290), "As with production functions (and to a lesser extent utility functions), finding ways to discriminate among functional forms empirically would be a complementary and welcome endeavor." We take this directly to heart shortly, specifying an additional linear CSF for analysis.

In (2), the parameter γ determines the "power" of talent on winning percent. If $\gamma = 0$, $w_1 = w_2 = 0.500$ and talent choice is irrelevant to the winning percent outcome for either team. For $\gamma > 0$, the behavior of winning percent with respect to talent depends on TSE. Taking the derivative in (2) yields MPT for the general logistic CSF:

$$(3)\quad \frac{dw_1}{dt_1} = z(t_1, t_2, \gamma)\left(t_2 - t_1\frac{dt_2}{dt_1}\right),\ \frac{dw_2}{dt_2} = z(t_1, t_2, \gamma)\left(t_1 - t_2\frac{dt_1}{dt_2}\right),$$

where $z(t_1, t_2, \gamma) = \dfrac{\gamma t_1^{\gamma-1} t_2^{\gamma-1}}{\left(t_1^{\gamma} + t_2^{\gamma}\right)^2}$.

The behavior of the marginal product of talent depends on TSE as before, that is, $\frac{dt_j}{dt_i}$ on the right-hand side of the MPTs in (3). We stress that TSE comes into play in the CSF directly before any consideration of owner objective functions and bears just as heavily on any choice of owner objective function by the modeler.

The most commonly used CSF in sports economics is the logistic in (2) with $\gamma = 1$. In this special case, all units of talent have equal power so that winning percent is simply determined by a team's relative share of total league talent, that is, $w_1 = \frac{t_1}{t_1 + t_2}$, $w_2 = \frac{t_2}{t_1 + t_2}$, and $w_2 = 1 - w_1$ from expression (2). Note that in this case $z(t_1, t_2, \gamma) = \frac{1}{(t_1 + t_2)^2}$ in (3).

Harkening back to the encouragement by Skaperdas (1996), our second CSF is a linear version as follows:

(4) $w_1 = a + t_1 - t_2$, $w_2 = 1 - w_1$, with $0 \le a \le 1 - (t_1 - t_2)$.

In the linear CSF in (4), team 1 makes headway against team 2 starting from a base winning percent given by the parameter “a” only in terms of the net increase in talent over team 2 (and vice versa). The second part of the linear CSF follows from the adding-up constraint and the remaining condition on the constant term follows from $0 \le w_1 \le 1$.

Taking the derivative of (4) yields MPT for the linear CSF:

(5) $\frac{dw_1}{dt_1} = 1 - \frac{dt_2}{dt_1}$, $\frac{dw_2}{dt_2} = 1 - \frac{dt_1}{dt_2}$.

Note in (5) that $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2}$ since $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2}$ for either closed or open leagues. As with the logistic CSF, the marginal product for the linear CSF depends on TSE, that is, $\frac{dt_j}{dt_i}$ on the right-hand side of the MPT functions in (5). However, the behavior of MPT under the linear CSF is relatively simple compared to the logistic case since there is no power parameter like γ in (5).

Our final element of analysis is a characterization of the league's profit-maximization equilibrium. The equal proportion revenue sharing profit functions for the two teams are given by:

$$(6)\quad \pi_1 = \alpha R_1(w_1(t_1, t_2(t_1))) + (1-\alpha)R_2(w_2(t_2(t_1), t_1)) - c\,t_1,$$

$$\pi_2 = \alpha R_2(w_2(t_2, t_1(t_2))) + (1-\alpha)R_1(w_1(t_1(t_2), t_2)) - c\,t_2,$$

where α is the home team's share of revenue, R_i is revenue generated when team *i* plays at location *i*. Since ours is a competitive talent market, the price that generates the market clearing constant marginal cost of talent "c" is generated by the usual Walrasian tatonnement process. Taking into account the implications of the adding-up constraint so that $\frac{dw_1}{dt_1} = -\frac{dw_2}{dt_1}$ and $\frac{dw_2}{dt_2} = -\frac{dw_1}{dt_2}$, the first-order conditions are:

$$(7)\quad \frac{d\pi_1}{dt_1} = \alpha\frac{dR_1}{dw_1}\left(\frac{\partial w_1}{\partial t_1} - \frac{\partial w_2}{\partial t_2}\frac{dt_2}{dt_1}\right) + (1-\alpha)\frac{dR_2}{dw_2}\left(\frac{\partial w_2}{\partial t_2}\frac{dt_2}{dt_1} - \frac{\partial w_1}{\partial t_1}\right) - c = 0,$$

$$\frac{d\pi_2}{dt_2} = \alpha\frac{dR_2}{dw_2}\left(\frac{\partial w_2}{\partial t_2} - \frac{\partial w_1}{\partial t_1}\frac{dt_1}{dt_2}\right) + (1-\alpha)\frac{dR_1}{dw_1}\left(\frac{\partial w_1}{\partial t_1}\frac{dt_1}{dt_2} - \frac{\partial w_2}{\partial t_2}\right) - c = 0.$$

In (7), marginal revenue equals marginal cost for each team owner. In equilibrium, this means that marginal revenues are equal to each other. Rearranging terms, equilibrium is characterized by:

$$(8)\quad \left[\alpha\frac{dR_1}{dw_1}-(1-\alpha)\frac{dR_2}{dw_2}\right]\left(\frac{\partial w_1}{\partial t_1}-\frac{\partial w_2}{\partial t_2}\frac{dt_2}{dt_1}\right)$$

$$=\left[\alpha\frac{dR_2}{dw_2}-(1-\alpha)\frac{dR_1}{dw_1}\right]\left(\frac{\partial w_2}{\partial t_2}-\frac{\partial w_1}{\partial t_1}\frac{dt_1}{dt_2}\right).$$

In the case of modern pooled sharing, $(1-\alpha')(R_1+R_2)$ comprises the pool shared equally by both owners. For our framework, profits for each team and the equilibrium condition for modern pooled sharing become:

$$(9)\quad \pi_1=\alpha' R_1(w_1(t_1,t_2(t_1)))$$

$$+\frac{(1-\alpha')}{2}\left[R_1(w_1(t_1,t_2(t_1)))+R_2(w_2(t_2(t_1),t_1))\right]-c\,t_1,$$

$$\pi_2=\alpha' R_2(w_2(t_2,t_1(t_2)))t_2$$

$$+\frac{(1-\alpha)}{2}\left[R_2(w_2(t_2,t_1(t_2)))+R_1(w_1(t_1(t_2),t_2))\right]-c\,t_2.$$

In the two-team league, pooled sharing in (9) adds more weight to each owner's own marginal revenue from winning percent, compared to the equal proportion sharing case in (6). However, if we let $\alpha=\frac{1+\alpha'}{2}$, then modern pooled sharing (mathematically) is completely analogous to the traditional equal proportion revenue sharing. As a result, all of the results that follow hold for both equal proportion and modern pooled revenue sharing.

The conditions under which equation (8) is independent of α gives us the conditions under which Rottenberg's IP holds for either the equal proportion or modern pooled revenue sharing case:

$$(10)\quad \left(\frac{\partial w_1}{\partial t_1} - \frac{\partial w_2}{\partial t_2}\frac{dt_2}{dt_1}\right) = \left(\frac{\partial w_2}{\partial t_2} - \frac{\partial w_1}{\partial t_1}\frac{dt_1}{dt_2}\right).$$

To see this, if (10) holds then the bracket terms in (8) are equal to each other and after simple algebra (8) reduces to $\frac{dR_1}{dw_1} = \frac{dR_2}{dw_2}$, independent of α, so that the IP holds. Thus, in the following sections, we will be examining conditions on TSE and the CSF under which (10) holds.

III. The General Case

First, we examine the IP in a "state of nature" without any assumptions whatsoever about TSE, UMPC, or the CSF. In such a "state of nature" (all proofs are in the Appendix and Table 1 summarizes the results of all of our propositions and corollaries that follow):

> <u>*Proposition 1 (P1)*</u>*: In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, without any assumption about the TSE, UMPC, or the CSF, there is no IP.*

P1 follows from the discussion at the end of the last section and seems trivial. But we include it because it does provide an important benchmark. Any insight into the IP, one of the most important theoretical findings in sports economics, actually does require *some* economic assumptions.

Moving on to those assumptions, our first finding is the following for closed and open leagues:

> Proposition 2 (P2): *In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, without any assumption about UMPC or the CSF, the IP holds for a closed league.*

> Proposition 3 (P3): *In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, without any assumption about the UMPC or the CSF, the IP does not hold for the open league.*

At our most general level, then, the interesting question from the perspective of the IP is whether the league is closed or open. P2 and P3 do two things relative to the past literature. First, for competitive talent markets, all that is needed to generate the IP is a closed league. Nobody to our knowledge has thought about these results being completely independent of UMPC and the form of the CSF. Second, while all others have found this result (and the related result cast in terms of conjectures in Szymanski, 2004) for the CSF in (2) with $\gamma = 1$, our P1 and P2 are completely general.

We now move on to the role of a recurrent but lightly assessed, assumption in the literature, namely, the UMPC:

> Proposition 4 (P4): *In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, without any assumption about TSE or the CSF, the IP holds for the UMPC.*

P4 is truly startling relative to the past literature. Without any assumptions about anything else, *the UMPC is enough on its own to generate the IP for both the closed and open league*. Nobody has noticed this power of the UMPC. Now, since the UMPC is essentially the assumption that MPT = 1, the question of interest is why the modeler expects unitary MPT.

This leads naturally to questions about the relationship between closed leagues, open leagues, and the UMPC and we find the following:

> <u>Proposition 5 (P5)</u>: *In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, without any assumption about the CSF, the UMPC is neither necessary nor sufficient for either a closed league or an open league.*

P5 clinches that the UMPC is separate and unequal to the assumption of a closed league. We can get the IP by either the UMPC or the closed league but using both is not necessary unless the modeler really wants MPT = 1 in a closed league. The UMPC is also independent of the open league but assuming it holds for an open league does generate the IP. Again, the crucial issue is why the modeler would need the separate restriction that MPT = 1. We note that the UMPC assumes that talent always has the same effect on winning. The closed league assumption assumes that increasing talent for one team decreases talent for the other team by the same amount. These are completely separate assumptions that both generate the IP.

IV. TSE, the Logistic CSF, and the IP

Our approach in this section is to examine the relationship between closed and open leagues and the IP under the general logistic CSF in (2) and its restricted version with $\gamma = 1$. Since we already have completely general results for the closed league and the IP (P2), and for the UMPC and the IP (P4), we are left to re-examine P3 and P5 under this particular CSF. For P3, we find:

> <u>Corollary 1, Proposition 3 (C1P3)</u>: *In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, without any assumption about the UMPC, under the general logistic CSF there is no IP for the open league.*

So, nothing is gained relative to the IP under this particular CSF, comparing P3 to C1P3, and the modeler does just as well without being restricted to the general logistic CSF.

For P5 we find:

<u>Corollary 1, Proposition 5 (C1P5)</u>: In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, under the general logistic CSF, the UMPC is sufficient but not necessary for the closed league; the UMPC is neither necessary nor sufficient for the open league.

For the closed league, C1P5 tells us that the UMPC implies that it is a closed league. This is different than the general case but, again, comes with the baggage of unit MPT.

Turning to the rest of the possibilities for the logistic CSF, we find the following for the open league when $\gamma = 1$:

<u>Corollary 2, Proposition 3 (C2P3)</u>: In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, under the logistic CSF with $\gamma = 1$, there is no IP for the open league.

<u>Corollary 2, Proposition 5 (C2P5)</u>: In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion revenue sharing or pooled sharing, under the logistic CSF with $\gamma = 1$, the UMPC is sufficient and necessary for the closed league but the UMPC is neither necessary nor sufficient for the open league.

C2P3 and C2P5 show that with the restriction $\gamma = 1$, the UMPC is now necessary as well as sufficient for the closed league. Because the restricted logistic model with $\gamma = 1$ has dominated the literature, C2P5 helps explain why the two assumptions have been used interchangeably. However, the two are not equivalent assumptions in the general case of the last section.

An interim summary is as follows. We generate the following new results. First, past results are extended to the general case that the IP holds for the closed league (P2) but not for the open league (P3). Second, the IP holds for the UMPC alone for any league in the general case (P4, P5). Third, all of the above hold for the general logistic CSF; to date the literature contains similar findings but only for the restricted case where

$\gamma = 1$ (C1P3, C1P5). Finally, for the sake of completeness, the UMPC is necessary and sufficient for the closed league under the logistic CSF with $\gamma = 1$ (C2P5).

V. TSE, the Linear CSF, and the IP

The linear CSF in (3) expands the possibilities for relationships between TSE and the IP. Again, following down our list of generality, only P3 and P5 remain open possibilities for the linear CSF. Relative to P3, we find:

> *Corollary 3, Proposition 3 (C3P3): In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, under the linear CSF, the IP occurs for the open league.*

This somewhat surprising result serves to emphasize that the existence of the IP cannot only be about whether a league is closed or open. The existence of the IP, instead, also depends on the CSF itself. We hasten to add that other baggage comes with C3P3 since the linear CSF necessarily has constant MPT, surely an empirical matter.

Finally, turning to P5, we are able to determine:

> *Corollary 3, Proposition 5 (C3P5): In a two team league of profit-maximizing owners with a competitive talent market, for either equal proportion or pooled revenue sharing, under the linear CSF, the UMPC is neither necessary nor sufficient for the closed league; the UMPC is both necessary and sufficient for the open league.*

Relative to our preceding results, this is quite surprising as well. But essentially we get C3P5 because MPT collapses to a constant in the case of the linear CSF and the UMPC has constant $\text{MPT} = 1$. Again, the result only serves to reinforce our basic observation that modeling choices make a difference in model results. The impact of the UMPC depends not only on a particular modeling choice (like closed or open leagues) but under some conditions on all modeling choices (like the choice of the CSF, itself).

VI. Conclusions

The competitive talent market literature analyzing Rottenberg's (1956) invariance proposition has failed to notice the importance of the relationship between three crucial elements and the invariance proposition, namely, talent supply elasticity, a common assumption setting the marginal product of talent equal to one, and the underlying contest success function. Choices concerning *both* talent supply elasticity *and* the contest success function generate different behavior of the marginal product of talent. As a result, it is these assumptions that determine differences in the profit-maximizing equilibrium and the presence of the invariance proposition. Put another way, whether a league is closed or open is only one of the modeling elements determining the existence of the invariance proposition.

Our findings include, first, that the invariance proposition characterizes the most general statement of the closed league but not the open league. Past work determined this for a particular contest success function, but it holds regardless of the contest success function. Our second result is that the unit marginal product of talent assumption is extremely strong, supporting the invariance proposition all by itself for both closed and open leagues. Finally, the invariance proposition does hold for an open league under an alternative linear contest success function and the unit measurement convention also is both necessary and sufficient for the open league under this contest success function.

To us, the lesson is that the modeler must decide why any particular choice of talent supply elasticity, unit marginal product of talent, and contest success function, yielding very predictable behavior of the marginal product of talent, best characterizes the league under analysis. Since these factors all impact the marginal product of talent, perhaps an assessment of their appropriateness for the modeling purpose is more important than identifying whether or not they support the invariance proposition.

And there is at least this for the empirical implications of our findings. The invariance proposition holds in general for a closed league with a competitive talent market. If the data reject the invariance proposition, this surely is a rejection of the closed league version of a competitive talent market and the particular behavior of the marginal product of talent associated with same. On the other hand, generally, the invariance proposition does not hold for an open league with a competitive talent market. But if the data end up consistent with the invariance proposition, the modeler *need not* reject an open league with a competitive talent market! An alternative explanation could be that the contest success function operational in the league under empirical scrutiny is of our alternative linear form, or some other contest success function not yet examined.

Our results suggest extreme caution in all sports modeling assumptions since they can be so important. In addition to careful attention to talent supply elasticity, there must be careful attention to the contest success function. It is generally invalid to argue that the invariance proposition holds for a particular talent supply elasticity while it does not hold for some other elasticity without identifying the contest success function as well. More important, to the extent that capturing real-world phenomena is one point of a particular analysis, talent supply elasticity and the contest success function should be specified to match the type of league under consideration. Eckard (2006), Fort and Quirk (2007), and Fort (2007) argue that North American leagues have cooperative settings consistent with closed leagues while other world leagues and North American college sports may be more consistent with open leagues. It is not enough, given the developments in the literature so far, to argue model superiority solely on the basis of beliefs about one type of talent supply or another.

Extensions include the research areas omitted intentionally in this paper (but cited extensively)— situations where fans care about absolute levels of talent and extensions to incorporate the relative size of revenue function shifts in smaller- and larger-revenue markets. Further, moving this literature in the direction of strategic choices in the talent market will prove enlightening for markets characterized by such behavior. And it is essential in those developments that the issues covered in this paper receive attention in those endeavors. In addition, it is our sincerest belief that those more adept at specifying contest success functions in addition to the ones analyzed in this paper will doubtless greatly enhance our understanding of the invariance proposition. Ultimately, since the assumptions addressed in this paper all determine the marginal product of talent, empirical work assessing that important production characteristic should prove definitive in guiding future theoretical work on sports leagues.

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Appendix: Proofs of Propositions.

Proof of Proposition 1

Without any assumptions at all about TSE, the CSF, or the UMPC, expression (10) fails to hold; that is, $\left(\frac{\partial w_1}{\partial t_1} - \frac{\partial w_2}{\partial t_2}\frac{dt_2}{dt_1}\right) \neq \left(\frac{\partial w_2}{\partial t_2} - \frac{\partial w_1}{\partial t_1}\frac{dt_1}{dt_2}\right)$ and equilibrium in (8) still depends on α so there is no IP.

Proof of Proposition 2

Fixed talent supply means that TSE = 0, implying $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2} = -1$. Substituting into (10) implies the IP iff $\left(\frac{\partial w_1}{\partial t_1} + \frac{\partial w_2}{\partial t_2}\right) = \left(\frac{\partial w_2}{\partial t_2} + \frac{\partial w_1}{\partial t_1}\right)$ which is identically true. As a result, the equilibrium in (8) does not depend on α and the IP holds.

Proof of Proposition 3

Perfectly elastic talent supply means that $\text{TSE} = \infty$, implying $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2} = 0$.

Substituting into (10) implies that the IP holds iff $\frac{\partial w_1}{\partial t_1} = \frac{\partial w_2}{\partial t_2}$ which need not hold generally. As a result, the equilibrium in (8) still depends on α so the IP does not hold.

Proof of Proposition 4

The UMPC has $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2} = 1$. Under this condition, substituting into (10) simply gives us $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2}$ which is true for either closed or open leagues. As a result, the equilibrium in (8) does not depend on α so the IP holds.

<u>*Proof of Proposition 5*</u>

If we assume the UMPC, then $\left(\frac{\partial w_1}{\partial t_1} - \frac{\partial w_2}{\partial t_2}\frac{dt_2}{dt_1}\right) = \left(\frac{\partial w_2}{\partial t_2} - \frac{\partial w_1}{\partial t_1}\frac{dt_1}{dt_2}\right) = 1$. This implies that $\frac{dt_2}{dt_1} = \frac{\frac{\partial w_1}{\partial t_1} - 1}{\frac{\partial w_2}{\partial t_2}}$ and $\frac{dt_1}{dt_2} = \frac{\frac{\partial w_2}{\partial t_2} - 1}{\frac{\partial w_1}{\partial t_1}}$. Therefore, $\frac{dt_2}{dt_1}$ and $\frac{dt_1}{dt_2}$ do not have to equal 0 or -1.

If we assume a closed league, $\frac{dw_1}{dt_1} = \left(\frac{\partial w_1}{\partial t_1} + \frac{\partial w_2}{\partial t_2}\right)$ and $\frac{dw_2}{dt_2} = \left(\frac{\partial w_1}{\partial t_1} + \frac{\partial w_2}{\partial t_2}\right)$, which do not necessarily equal 1. We also note that $\frac{dw_1}{dt_1}$ and $\frac{dw_2}{dt_2}$ do not have to equal a constant. If we assume an open league, $\frac{dw_1}{dt_1} = \frac{\partial w_1}{\partial t_1}$ and $\frac{dw_2}{dt_2} = \frac{\partial w_2}{\partial t_2}$. Again these do not have to equal one.

<u>*Proof of Corollary 1, Proposition 3*</u>

In the proof of Proposition 3, for the open league, under the general case, we get the IP <u>iff</u> $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2}$. Using equation (3), $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2}$ <u>iff</u>

$z(t_1, t_2, \gamma)\left(t_2 - t_1\frac{dt_2}{dt_1}\right) = z(t_1, t_2, \gamma)\left(t_1 - t_2\frac{dt_1}{dt_2}\right)$. For the open league, $\frac{dt_1}{dt_2} = \frac{dt_2}{dt_1} = 0$ and equality reduces to $t_1 = t_2$. This holds only trivially for the perfectly balanced league. Restricting to the general logistic CSF does not gain the IP for the open league.

Proof of Corollary 1, Proposition 5

If we assume the UMPC for the general logistic CSF, then

$z(t_1, t_2, \gamma)\left(t_2 - t_1 \frac{dt_2}{dt_1}\right) = z(t_1, t_2, \gamma)\left(t_1 - t_2 \frac{dt_1}{dt_2}\right) = 1$, which reduces to

$t_2 - t_1 \frac{dt_2}{dt_1} = t_1 + t_2 \frac{dt_1}{dt_2}$. Since the TSE should be the same for both teams, the equality only holds when $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2} = -1$, implying a closed league. If we assume a closed league, $\frac{dw_1}{dt_1} = z(t_1, t_2, \gamma)(t_2 + t_1)$ and $\frac{dw_2}{dt_2} = z(t_1, t_2, \gamma)(t_1 + t_2)$, which do not necessarily equal 1, or a constant. We note that in a closed league $t_1 + t_2$ is a constant. However, $z(t_1, t_2, \gamma)$ is not restricted to be a constant and depends on the distribution of talent. This indicates that the UMPC is sufficient but not necessary for the closed league under the general logistic CSF

For the open league, the UMPC cannot be sufficient for the open league if it is sufficient for the closed league. In an open league, $\frac{dt_1}{dt_2} = \frac{dt_2}{dt_1} = 0$. If we assume an open league for the general logistic model, $\frac{dw_1}{dt_1} = z(t_1, t_2, \gamma)t_2$ and $\frac{dw_2}{dt_2} = z(t_1, t_2, \gamma)t_1$.

Therefore, $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2} = 1$ requires $t_1 = t_2 = \frac{1}{z(t_1, t_2, \gamma)}$. Since there is no reason for this to be true generally, the UMPC is not necessary for the open league. Overall, the UMPC is neither necessary nor sufficient for the open league.

Proof of Corollary 2, Proposition 3

In the proof of Proposition 3, for the open league, under the general case, we get the IP iff $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2}$. Using equation (3), $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2}$ iff

$z(t_1, t_2, \gamma)\left(t_2 - t_1 \frac{dt_2}{dt_1} \right) = z(t_1, t_2, \gamma)\left(t_1 - t_2 \frac{dt_1}{dt_2} \right)$ where $z(t_1, t_2, \gamma) = \frac{1}{(t_1 + t_2)^2}$. For the open league, $\frac{dt_1}{dt_2} = \frac{dt_2}{dt_1} = 0$ and equality reduces to $t_1 = t_2$. This holds only trivially for the perfectly balanced league. Restricting to the general logistic CSF does not gain the IP for the open league.

Proof of Corollary 2, Proposition 5

Because the UMPC is sufficient for a closed league with the general logistic CSF, it is also sufficient for the restricted CSF. We now check for the necessary condition. If we assume a closed league, $\frac{dw_1}{dt_1} = z(t_1, t_2, \gamma)(t_2 + t_1)$ and $\frac{dw_2}{dt_2} = z(t_1, t_2, \gamma)(t_2 + t_1)$ where $z(t_1, t_2, \gamma) = \frac{1}{(t_1 + t_2)^2}$. This reduces to $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2} = \frac{1}{t_2 + t_1}$. Because talent is fixed it can be normalized so that $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2} = 1$ and the UMPC holds.

For the open league, again, since the UMPC is sufficient for the closed league, it cannot be sufficient for the open league. Turning to the necessary condition, with $\gamma = 1$, $z(t_1, t_2, \gamma) = \frac{1}{(t_1 + t_2)^2}$ but once again $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2} = 1$ requires $t_1 = t_2 = \frac{1}{z(t_1, t_2, \gamma)}$. Since there is no reason for this to be true generally, the UMPC is not necessary for the open league. Overall, the UMPC is neither necessary nor sufficient for the open league in this more restrictive case.

Proof of Corollary 3, Proposition 3

Again, for the open league, we get the IP iff $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2}$. From (5), $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2} = 1$ since $\frac{dt_1}{dt_2} = \frac{dt_2}{dt_1} = 0$ for the open league. Thus, the IP does hold for the linear CSF for the open league.

Proof of Corollary 3, Proposition 5

First, examine conditions for the closed league. For sufficiency, substituting the UMPC into (5) yields $\frac{dw_1}{dt_1} = 1 - \frac{dt_2}{dt_1} = 1$ and $\frac{dw_2}{dt_2} = 1 - \frac{dt_1}{dt_2} = 1$. But together these imply $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2} = 0$, contradicting the definition of the closed league. The UMPC is not sufficient for the closed league. For the necessary condition, substituting the definition of the closed league into (5) yields $\frac{dw_1}{dt_1} = \frac{dw_1}{dt_1} = 1 - (-1) = 2$. But this contradicts the definition of the UMPC so the UMPC is not necessary for the closed league either. Overall, the UMPC is neither necessary nor sufficient for the closed league under the linear CSF. We note that there is no normalization of talent that provides the UMPC. In other words, the effect of talent on winning always has a greater effect under a closed league than with the UMPC.

For the open league, for sufficiency, substituting the UMPC into (5) yields $\frac{dw_1}{dt_1} = 1 - \frac{dt_2}{dt_1} = 1$ and $\frac{dw_2}{dt_2} = 1 - \frac{dt_1}{dt_2} = 1$. But together these imply $\frac{dt_2}{dt_1} = \frac{dt_1}{dt_2} = 0$, which is the definition of the open league. So, the UMPC is sufficient for the open league under the linear CSF. For the necessary condition, substitute the definition of the open league into (5) and immediately $\frac{dw_1}{dt_1} = \frac{dw_2}{dt_2} = 1 - 0 = 1$, the definition of the UMPC. So, the UMPC is necessary for the open league. Overall, the UMPC is both necessary and sufficient for the open league under the linear CSF.

Table 1. Summary of Propositions and Corollaries.

Closed League ($TSE = 0$)	Open League ($TSE = \infty$)
Generally	
IP (P2)	No IP (P3)
UMPC suff IP (P4)	UMPC suff IP(P4)
UMPC neither nec nor suff Closed (P5)	UMPC neither nec nor suff Open (P5)
General Logistic CSF	
IP (P2)	No IP (C1P3)
UMPC suff IP (P4)	UMPC suff IP(P4)
UMPC suff but not nec Closed (C1P5)	UMPC neither nec nor suff Open (C1P5)
Logistic CSF $\gamma = 1$	
IP (P2)	No IP (C2P3)
UMPC suff IP (P4)	UMPC suff IP(P4)
UMPC suff & nec Closed (C2P5)	UMPC neither nec nor suff Open (P5)
Linear CSF	
IP (P2)	IP (C3P3)
UMPC suff IP (P4)	UMPC suff IP(P4)
UMPC neither nec nor suff Closed (C3P5)	UMPC nec & suff Open (C3P5)

Notes: TSE = talent supply elasticity; IP = invariance proposition; UMPC = unit marginal product convention; nec = necessary and suff = sufficient; Pj denotes Proposition j, j = 1,…, 5; Ci denotes Corollary i, i = 1,…, 3.